Kinetic Equations

Solution to the Exercises

– 01.04.2021 –

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Exercise 1

Recall that for any f measurable the *distribution function of* f is defined as $d_f(\gamma) := \left|\left\{x \in \mathbb{R}^d \middle| \ |f(x)| > \gamma\right\}\right|$. Prove that if $f \in L^p\left(\mathbb{R}^d\right)$ with $p \in [1, +\infty)$ then

$$\|f\|_{L^p(\mathbb{R}^d)} = \left(\int_0^{+\infty} p\gamma^{p-1} d_f(\gamma)\, d\gamma\right)^{\frac{1}{p}}. \tag{1}$$

Proof. By definition of the distribution function, we get that

$$d_f(\gamma) = \int_{\{x \in \mathbb{R}^d | \ |f(x)| > \gamma\}} dx. \tag{2}$$

As a consequence we get

$$\begin{aligned} \int_0^{+\infty} p\gamma^{p-1} d_f(\gamma)\, d\gamma &= \int_0^{+\infty} p\gamma^{p-1} \int_{\{x \in \mathbb{R}^d | \ |f(x)| > \gamma\}} dx d\gamma & (3)\\ &= \int_{\mathbb{R}^d} \int_0^{|f(x)|} p\gamma^{p-1} d\gamma dx = \int_{\mathbb{R}^d} |f(x)|^p\, dx = \|f\|_{L^p(\mathbb{R}^d)}^p. & (4) \end{aligned}$$

□

Exercise 2

Let $m \geqslant 0$ and f a measurable function such that $f \geqslant 0$ and $f \in L^1 \cap L^\infty\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Define the measurable function $h : \mathbb{R} \times \mathbb{R}^3 \to [0, +\infty]$ as

$$h(t, x) := \int_{\mathbb{R}^3} f(x - tv, v)\, dv. \tag{5}$$

Show that there is a constant $C > 0$ such that

$$\|h(t, \cdot)\|_{L^{\frac{m+3}{3}}(\mathbb{R}^3)} \leqslant C \|f\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^m f(x, v)\, dx dv\right)^{\frac{3}{m+3}}. \tag{6}$$

Proof. Similarly at what we did in class, we first fix $R > 0$ and get

$$\begin{aligned} |h(t, x)| &= \int_{|x| \leqslant R} f(x - tv, v)\, dv + \int_{|x| > R} f(x - tv, v)\, dv & (7)\\ &\leqslant \frac{4\pi}{3} R^3 \|f\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} + \frac{1}{R^m} \int_{\mathbb{R}^3} |v|^m f(x - tv, v)\, dv. & (8) \end{aligned}$$

Optimizing on R we get

$$|h(t,x)| \leqslant \frac{m+3}{(27m^m)^{\frac{1}{m+3}}} \left(\frac{4\pi}{3}\|f\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\right)^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3} |v|^m f(x-tv,v)\,dv\right)^{\frac{3}{m+3}} \tag{9}$$

$$= C_m \|f\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3} |v|^m f(x-tv,v)\,dv\right)^{\frac{3}{m+3}}. \tag{10}$$

We then consider the $L^{\frac{m+3}{3}}$ norm of h to get

$$\|h(t,\cdot)\|_{L^{\frac{m+3}{m}}(\mathbb{R}^3)} \leqslant C_m \|f\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |v|^m f(x-tv,v)\,dv\right)dx\right)^{\frac{m}{m+3}} \tag{11}$$

$$\leqslant C_m \|f\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}^{\frac{m}{m+3}} \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^m f(x,v)\,dxdv\right)^{\frac{m}{m+3}}, \tag{12}$$

which concludes the proof of the result.

□

Exercise 3

Let $p \in [1,+\infty)$ and $f \in C\left(\mathbb{R}_+; L^p\left(\mathbb{R}^3\times\mathbb{R}^3\right)\right) \cap L^\infty\left(\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3\right)$ a solution to the Vlasov-Poisson equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E\cdot\nabla_v f = 0, \\ E(t,x) := -\nabla\left(\frac{1}{|x|} * \rho_t\right)(x), \\ \rho_t(x) := \int_{\mathbb{R}^3} f(t,x,v)\,dv, \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{13}$$

Define $H(t)$ as

$$H(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f(t,x,v)\,dxdv + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_t(x)\,\rho_t(x')}{|x-x'|}\,dxdx'. \tag{14}$$

Prove that $H(t) = H(0)$.

Hint: Show that $\partial_t H(t) = 0$.

Proof. First notice that the derivative of the density ρ_t is given by

$$\partial_t \rho_t(x) = \partial_t \int_{\mathbb{R}^3} f(t,x,v)\,dv = -\int_{\mathbb{R}^3} \left(v\cdot\nabla_x f(t,x,v) + E(t,x)\cdot\nabla_v f(t,x,v)\right)dv \tag{15}$$

$$= -\operatorname{div}_x \int_{\mathbb{R}^3} v f(t,x,v)\,dv. \tag{16}$$

In order to differentiate $H\left(t\right)$, we first get that the derivative of the kinetic energy is given by

$$\partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|v\right|^2}{2} f\left(t,x,v\right) dxdv = -\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left|v\right| \left(v\cdot\nabla_x f\left(t,x,v\right) + E\left(t,x\right)\cdot\nabla_v f\left(t,x,v\right)\right) dxdv \tag{17}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v\cdot E\left(t,x\right) f\left(t,x,v\right) dxdv \tag{18}$$

We then get

$$\partial_t H\left(t\right) = \partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|v\right|^2}{2} f\left(t,x,v\right) dxdv + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\partial_t\rho_t\left(x\right)\rho_t\left(x'\right)}{\left|x-x'\right|} dxdx' \tag{19}$$

$$+\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho_t\left(x\right)\partial_t\rho_t\left(x'\right)}{\left|x-x'\right|} dxdx' \tag{20}$$

$$= \partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|v\right|^2}{2} f\left(t,x,v\right) dxdv + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\partial_t\rho_t\left(x\right)\rho_t\left(x'\right)}{\left|x-x'\right|} dxdx' \tag{21}$$

$$= \partial_t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left|v\right|^2}{2} f\left(t,x,v\right) dxdv + \int_{\mathbb{R}^3} \frac{1}{\left|x\right|} * \rho_t\left(x\right)\partial_t\rho_t\left(x\right) dx \tag{22}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v\cdot E\left(t,x\right) f\left(t,x,v\right) dxdv \tag{23}$$

$$-\int_{\mathbb{R}^3} \frac{1}{\left|x\right|} * \rho_t\left(x\right) \mathrm{div}_x \int_{\mathbb{R}^3} v f\left(t,x,v\right) dvdx \tag{24}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} v\cdot E\left(t,x\right) f\left(t,x,v\right) dxdv \tag{25}$$

$$+\int_{\mathbb{R}^3} \nabla\left(\frac{1}{\left|x\right|} * \rho_t\right)\left(x\right)\cdot\int_{\mathbb{R}^3} v f\left(t,x,v\right) dvdx = 0. \tag{26}$$

As a consequence, if $\partial_t H\left(t\right) = 0$ this implies that the energy $H\left(t\right)$ is constant and therefore $H\left(t\right) = H\left(0\right)$.

$\square$